Consider a real symmetric matrix $P \in \mathbb{R}^{n\times n}$, a column vector $x \in \mathbb{R}^n$, and $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ are eigenvalues of P.
Prove that

$$\lambda_1 x^T x \le x^T P x \le \lambda_n x^T x$$

Proof:
Based on Theorem 3.6 (see Appendix), P can be written as

$$P = QVQ^T$$

where Q is an orthogonal matrix and V is a diagonal matrix. The diagonal elements of V are the eigenvalues of P, i.e., $V = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$.

Denote $Q = [q_1 \quad \cdots \quad q_n]$ and then

$$x^T P x = x^T Q V Q^T x = (Q^T x)^T V (Q^T x) = [(q_1^T x)^T \quad \cdots \quad (q_n^T x)^T] V \begin{bmatrix} q_1^T x \\ \vdots \\ q_n^T x \end{bmatrix}$$

$$= \sum_{i=1}^{n} \lambda_i \left(\left(q_i^T x\right)^T q_i^T x \right) = \sum_{i=1}^{n} \lambda_i \left(x^T q_i q_i^T x \right)$$

Note that $x^T q_i q_i^T x$ is a scalar and nonnegative, so it follows that

$$\sum_{i=1}^{n} \lambda_i \left(x^T q_i q_i^T x \right) \le \lambda_n \left(\sum_{i=1}^{n} x^T q_i q_i^T x \right) = \lambda_n x^T \left(\sum_{i=1}^{n} q_i q_i^T \right) x$$

$$= \lambda_n x^T \left([q_1 \quad \cdots \quad q_n] \begin{bmatrix} q_1^T \\ \vdots \\ q_n^T \end{bmatrix} \right) x = \lambda_n x^T (QQ^T) x$$

Since Q is the orthogonal matrix, $QQ^T = I$. Therefore, $x^T P x \le \lambda_n x^T x$. Similarly, we can obtain that $x^T P x \ge \lambda_1 x^T x$. This completes the proof.

Appendix:

Theorem 3.6

For every real symmetric matrix $\mathbf{M}$, there exists an orthogonal matrix $\mathbf{Q}$ such that

$$\mathbf{M} = \mathbf{Q}\mathbf{D}\mathbf{Q}' \quad \text{or} \quad \mathbf{D} = \mathbf{Q}'\mathbf{M}\mathbf{Q}$$

where $\mathbf{D}$ is a diagonal matrix with the eigenvalues of $\mathbf{M}$, which are all real, on the diagonal.

This is from a book named *Linear System Theory and Design*, third edition. The author is Chi-Tsong Chen.